

An almost all result on $q_1q_2 \equiv c \pmod{q}$

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Abstract

In this paper we consider the congruence equation $q_1q_2 \equiv c \pmod{q}$ with $a < q_1 \leq a + q^{1/2+\epsilon}$ and $b < q_2 \leq b + q^{1/2+\epsilon}$ and show that it has solution for almost all a and b . Then we apply it to a question of Fujii and Kitaoka as well as generalize it to more variables. At the end, we will present a new way to attack the above congruence equation question through higher moments.

1 Introduction and main results

A famous congruence equation question is the following:

Question 1 *Given any $\epsilon > 0$. Is it true that, for any modulus $q \geq 1$ and any integer c with $(c, q) = 1$, the congruence equation $q_1q_2 \equiv c \pmod{q}$ is solvable for some $1 \leq q_1, q_2 \ll_\epsilon q^{1/2+\epsilon}$?*

Davenport [2] used Kloosterman sum estimates to show that the above question is true for all $\epsilon > 1/3$. Using Weil's bound on Kloosterman sums (see equation (2)), Davenport's argument implies the truth of Question 1 for all $\epsilon > 1/4$. Recently in [11], Shparlinski got the same result with the further restriction that q_1, q_2 are relatively prime to one another. When q is a prime number, Garaev [6] obtained a slight improvement that Question 1 is true for all $\epsilon \geq 1/4$.

Question 1 seems to be hard. How about proving it for almost all c ? Recently Garaev and Karatsuba [8], and Shparlinski [12] proved that the above question is true for almost all c with any $\epsilon > 0$ when q is prime and q in general respectively. Their results are more general as one of the interval can be replaced by a sufficiently large subset of the interval and the other interval does not have to start from 1. Furthermore when q is prime, Garaev and Garcia [7] showed the above almost all result with q_1, q_2 in any intervals of length $q^{1/2+\epsilon}$ by considering solutions to $q_1q_2 \equiv q_3q_4 \pmod{q}$. It used both character sum technique of [1] and exponential sum technique of [6].

Thus, in general, one does not have to restrict the ranges of q_1 and q_2 to start from 1. In fact, the above question should be true for q_1 and q_2 in any interval of length $O_\epsilon(q^{1/2+\epsilon})$. In this paper, we will prove that this is indeed the case for almost all such pairs of intervals for q_1 and q_2 , namely

Theorem 1 *For any modulus $q \geq 1$ and any integers $1 \leq L \leq q$ and $(c, q) = 1$,*

$$S := \sum_{a=1}^q \sum_{b=1}^q \left(\sum_{\substack{q_1 \in (a, a+L] \\ q_1q_2 \equiv c \pmod{q}}} \sum'_{q_2 \in (b, b+L]} 1 - \frac{1}{q} \sum_{q_1 \in (a, a+L]} \sum'_{q_2 \in (b, b+L]} 1 \right)^2 \ll L^2 q d(q)^3$$

where \sum' means summing over those numbers that are relatively prime to q .

Let us interpret Theorem 1. Since

$$\sum'_{q_2 \in (b, b+L]} 1 = \frac{\phi(q)}{q} L + O(d(q)),$$

we have

$$\begin{aligned} \frac{1}{q} \sum_{q_1 \in (a, a+L]} \sum'_{q_2 \in (b, b+L]} 1 &= \frac{\phi(q)}{q^2} L^2 + O\left(\frac{d(q)L}{q} + \frac{L}{q^{1/2-\epsilon/2}}\right) \\ &= \frac{\phi(q)}{q^2} L^2 + O\left(\frac{L}{q^{1/2-\epsilon/2}}\right) \end{aligned}$$

by $d(q) \ll q^{\epsilon/2}$. Note that the error term is smaller than the main term when $q^{1/2+\epsilon} \ll L$. Thus, if we let N be the number of pairs of (a, b) such that $q_1 q_2 \equiv c \pmod{q}$ has no solution with $q_1 \in (a, a+L]$ and $q_2 \in (b, b+L]$, then

$$N \left(\frac{\phi(q)}{q^2} L^2 \right)^2 \ll L^2 q d(q)^3.$$

This implies $N \ll q^{3+\epsilon}/L^2 \leq q^{2-\epsilon}$ using $\phi(q) \gg q/\log \log q$ and $L \gg q^{1/2+\epsilon}$. Consequently, with $L = C_\epsilon q^{1/2+\epsilon}$ where $C_\epsilon > 0$ is large enough, we have

Corollary 1 *Given any modulus $q \geq 1$ and any integer c with $(c, q) = 1$. For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$q_1 q_2 \equiv c \pmod{q}$$

is solvable in $q_1 \in (a, a + C_\epsilon q^{1/2+\epsilon}]$, $q_2 \in (b, b + C_\epsilon q^{1/2+\epsilon}]$ for almost all pairs of a and b .

Interestingly, Theorem 1 can recover the currently best result to Question 1:

Corollary 2 *Given any modulus $q \geq 1$ and any integer c with $(c, q) = 1$. For any $\epsilon > 0$ and integers a and b , there exists a constant $C_\epsilon > 0$ such that*

$$q_1 q_2 \equiv c \pmod{q}$$

is solvable for some $q_1 \in (a, a + C_\epsilon q^{3/4+\epsilon}]$, $q_2 \in (b, b + C_\epsilon q^{3/4+\epsilon}]$.

This opens up a new way to attack Question 1 through looking at higher moment analogue of Theorems 1 or 3. We shall discuss this in the last section.

In a similar spirit, Fujii and Kitaoka [4] studied the

Question 2 *For any lattice point $(x, y) \in \mathbb{Z}^2$, let $C_{(x,y)}(r)$ denote the compact disc with center (x, y) and radius r . Let q be a large number. Find the infimum $r(q)$ of all real numbers r such that the square $[0, q] \times [0, q]$ is covered by*

$$\bigcup_{\substack{x, y=1 \\ xy \equiv 1 \pmod{q}}}^{q-1} C_{(x,y)}(r).$$

When q is a prime number, they proved that $r(q) \ll q^{3/4} \log q$ and conjectured that $r(q) \ll_\epsilon q^{1/2+\epsilon}$ for every $\epsilon > 0$. Garaev [5] mentioned that the argument of [9] gives $r(q) \ll q^{3/4}$ for prime q . Using Theorem 1, we can answer Question 2 in an almost all sense:

Corollary 3 *With the notations in Question 2, let $\tilde{r}(q)$ be the infimum of all real numbers r such that*

$$\bigcup_{\substack{x,y=1 \\ xy \equiv 1 \pmod{q}}}^{q-1} C_{(x,y)}(r)$$

cover an area of $(1 + o(1))q^2$ in the square $[0, q] \times [0, q]$. Then $\tilde{r}(q) \ll_\epsilon q^{1/2+\epsilon}$ for any $\epsilon > 0$.

By modifying the proof of Theorem 1 slightly, one can get

Theorem 2 *For any modulus $q \geq 1$ and any integers $1 \leq L_1, L_2 \leq q$ and $(c, q) = 1$,*

$$\sum_{a=1}^q \sum_{b=1}^q \left(\sum_{\substack{q_1 \in (a, a+L_1] \\ q_1 q_2 \equiv c \pmod{q}}} \sum'_{q_2 \in (b, b+L_2]} 1 - \frac{1}{q} \sum_{q_1 \in (a, a+L_1]} \sum'_{q_2 \in (b, b+L_2]} 1 \right)^2 \ll L_1 L_2 q d(q)^3$$

where \sum' means summing over those numbers that are relatively prime to q .

Then one can discuss the above results for rectangles and eclipses instead of squares and circles. We leave these for the readers to explore.

More generally, one can consider the more variable version:

$$q_1 q_2 \dots q_t \equiv c \pmod{q}. \quad (1)$$

Using character sum method, Shparlinski and Winterhof [13] recently proved that for any $\epsilon > 0$ and $(c, q) = 1$,

$$q_1 q_2 q_3 \equiv c \pmod{q} \text{ is solvable for some } 1 \leq q_1, q_2, q_3 \leq q^{2/3+\epsilon}$$

and, for $t \geq 4$,

$$q_1 q_2 \dots q_t \equiv c \pmod{q} \text{ is solvable for some } 1 \leq q_1, q_2, \dots, q_t \leq q^{1/3+1/(t+2)+\epsilon}.$$

We shall imitate Theorem 1 and prove

Theorem 3 *For any modulus $q \geq 1$ and any integers $1 \leq L \leq q$ and $(c, q) = 1$,*

$$\begin{aligned} S &:= \sum_{a_1=1}^q \dots \sum_{a_t=1}^q \left| \sum'_{\substack{q_1 \in (a_1, a_1+L] \\ q_1 \dots q_t \equiv c \pmod{q}}} \dots \sum'_{q_t \in (a_t, a_t+L]} 1 - \left(\frac{L}{q}\right)^t \phi(q)^{t-1} \right|^2 \\ &\leq C_q^2 t^{2\omega(q)} q^{t-1} L^t d(q)^t \left[1 + \frac{t(L+1)^{t-1}}{q} \right] \end{aligned}$$

where $C_q = 1$ if q is odd and $C_q = 2^{(t+1)/2}$ if q is even.

Corollary 4 *Given any modulus $q \geq 1$ and any integer c with $(c, q) = 1$. For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$q_1 q_2 \dots q_t \equiv c \pmod{q}$$

is solvable in $q_1 \in (a_1, a_1 + C_\epsilon q^{1/t+\epsilon}]$, $q_2 \in (a_2, a_2 + C_\epsilon q^{1/t+\epsilon}]$, ..., $q_t \in (a_t, a_t + C_\epsilon q^{1/t+\epsilon}]$ for almost all t -tuples a_1, \dots, a_t .

The exponent $1/t + \epsilon$ is best possible. One may then imitate Corollary 2 and get a non-almost all result for (1). By averaging over c , one can show that $S \gg q^{t-1} L^t$ for some c . Thus, even with the best possible upper bound $O(q^{t-1} L^t)$ for Theorem 3, one can only prove that (1) has solution for q_1, \dots, q_t in intervals of length $q^{(t+1)/(2t)}$. These are no better than Shparlinski and Winterhof's results. So passing from our almost all result to non-almost all result is not a good approach unless one considers higher moments or can somehow generate more tuples of intervals without a solution out of a single one.

In summary, the method to study the above questions falls into two categories. One uses exponential sums, particularly Kloosterman and hyper-Kloosterman sums. The other one is character sum techniques including Polya-Vinogradov and Burgess bounds as well as fourth moment estimates on character sums (see [3] and [1]). It seems that character sum does better when there are more variables. However, for our almost all results, we shall use Kloosterman and hyper-Kloosterman sums.

The paper is organized as follows. First we will prove Theorem 1. The reason we do this first is that it is how this research began and it illustrates the essence of techniques used. Then we will prove Corollaries 2 and 3. After these, we will prove the general case, Theorem 3 and Corollary 4, more neatly using the language of finite Fourier series. Finally we will discuss higher moment attack of Question 1.

Notations Throughout the paper, ϵ denotes a small positive number. $f(x) \ll g(x)$ means that $|f(x)| \leq Cg(x)$ for some constant $C > 0$ and $f(x) \ll_\lambda g(x)$ means that the implicit constant $C = C_\lambda$ may depend on the parameter λ . Also $\phi(n)$ is Euler's phi function, $d(n)$ is the number of divisors of n and $\omega(n)$ is the number of distinct prime divisors of n .

2 Theorem 1

For $(q_2, q) = 1$, the congruence equation $q_1 q_2 \equiv c \pmod{q}$ is equivalent to $q_1 \equiv c \overline{q_2} \pmod{q}$ where \overline{n} denotes the multiplicative inverse of $n \pmod{q}$. By the orthogonal property of $e(u) = e^{2\pi i u}$,

$$\frac{1}{q} \sum_{a=1}^q e\left(\frac{ka}{q}\right) = \begin{cases} 1, & \text{if } k \equiv 0 \pmod{q}, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$S = \sum_{a=1}^q \sum_{b=1}^q \left(\frac{1}{q} \sum_{k=1}^{q-1} \sum_{q_1 \in (a, a+L]} \sum'_{q_2 \in (b, b+L]} e\left(\frac{k(q_1 - c\overline{q_2})}{q}\right) \right)^2.$$

Expanding things out, we have

$$\begin{aligned}
S &= \sum_{a=1}^q \sum_{b=1}^q \frac{1}{q^2} \sum_{k=1}^{q-1} \sum_{l=1}^{q-1} \sum_{q_1 \in (a, a+L]} \sum'_{q_2 \in (b, b+L]} e\left(\frac{k(q_1 - c\bar{q}_2)}{q}\right) \\
&\quad \times \sum_{q_3 \in (a, a+L]} \sum'_{q_4 \in (b, b+L]} e\left(\frac{-l(q_3 - c\bar{q}_4)}{q}\right) \\
&= \sum_{a=1}^q \sum_{b=1}^q \frac{1}{q^2} \sum_{k=1}^{q-1} \sum_{l=1}^{q-1} \sum_{q_1, q_3 \in (a, a+L]} \sum'_{q_2, q_4 \in (b, b+L]} e\left(\frac{kq_1}{q}\right) e\left(\frac{-lq_3}{q}\right) e\left(\frac{-kc\bar{q}_2}{q}\right) e\left(\frac{lc\bar{q}_4}{q}\right).
\end{aligned}$$

By making a change of variable $q_1 = q_3 + s$ with $-L \leq s \leq L$ and combining the sums over q_1 , q_3 and a , we have

$$\begin{aligned}
S &= \frac{1}{q^2} \sum_{k=1}^{q-1} \sum_{l=1}^{q-1} \sum_{q_3=1}^q e\left(\frac{(k-l)q_3}{q}\right) \sum_{s=-L}^L (L - |s|) e\left(\frac{ks}{q}\right) \\
&\quad \times \sum_{b=1}^q \sum'_{q_2, q_4 \in (b, b+L]} e\left(\frac{-kc\bar{q}_2}{q}\right) e\left(\frac{lc\bar{q}_4}{q}\right) \\
&= \frac{1}{q} \sum_{k=1}^{q-1} \sum_{s=-L}^L (L - |s|) e\left(\frac{ks}{q}\right) \sum_{b=1}^q \sum'_{q_2 \in (b, b+L]} e\left(\frac{-kc\bar{q}_2}{q}\right) \sum'_{q_4 \in (b, b+L]} e\left(\frac{kc\bar{q}_4}{q}\right) \\
&= \frac{1}{q} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}}\right)^2 \sum_{b=1}^q \sum'_{q_2 \in (b, b+L]} e\left(\frac{-kc\bar{q}_2}{q}\right) \sum'_{q_4 \in (b, b+L]} e\left(\frac{kc\bar{q}_4}{q}\right)
\end{aligned}$$

by Fejér kernel formula $\left(\frac{\sin \pi Nx}{\sin \pi x}\right)^2 = \sum_{j=-N}^N (N - |j|) e(jx)$. The innermost sums are incomplete Kloosterman sums. We can use standard technique to complete the sums:

$$\begin{aligned}
\sum'_{q_2 \in (b, b+L]} e\left(\frac{-kc\bar{q}_2}{q}\right) &= \sum'_{q_2=1}^q e\left(\frac{-kc\bar{q}_2}{q}\right) \sum_{m \in (b, b+L]} \frac{1}{q} \sum_{l=1}^q e\left(\frac{l(m - q_2)}{q}\right) \\
&= \frac{1}{q} \sum_{l=1}^q \sum_{m \in (b, b+L]} e\left(\frac{lm}{q}\right) \sum'_{q_2=1}^q e\left(\frac{-lq_2 - kc\bar{q}_2}{q}\right) \\
&= \frac{1}{q} \sum_{l=1}^q \left[\sum_{m \in (b, b+L]} e\left(\frac{lm}{q}\right) \right] S(-l, -kc; q)
\end{aligned}$$

where $S(a, b; q) := \sum'_{1 \leq n \leq q} e\left(\frac{an+b\bar{n}}{q}\right)$ is the Kloosterman sum. Therefore,

$$\begin{aligned}
S &= \frac{1}{q^3} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}}\right)^2 \sum_{l_1=1}^q \sum_{l_2=1}^q \sum_{b=1}^q \left[\sum_{m_1 \in (b, b+L]} e\left(\frac{l_1 m_1}{q}\right) \right] \\
&\quad \times \left[\sum_{m_2 \in (b, b+L]} e\left(\frac{-l_2 m_2}{q}\right) \right] S(-l_1, -kc; q) \overline{S(-l_2, -kc; q)}.
\end{aligned}$$

By making a change of variable $m_1 = m_2 + d$ with $-L \leq d \leq L$, the sums over m_1 , m_2 and b combine to give

$$\begin{aligned}
S &= \frac{1}{q^3} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}} \right)^2 \sum_{l_1=1}^q \sum_{l_2=1}^q \sum_{d=-L}^L (L - |d|) e\left(\frac{l_1 d}{q}\right) \sum_{m_2=1}^q e\left(\frac{(l_1 - l_2)m_2}{q}\right) \\
&\quad \times S(-l_1, -kc; q) \overline{S(-l_2, -kc; q)} \\
&= \frac{1}{q^2} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}} \right)^2 \sum_{l_1=1}^q \sum_{d=-L}^L (L - |d|) e\left(\frac{l_1 d}{q}\right) S(-l_1, -kc; q) \overline{S(-l_1, -kc; q)} \\
&= \frac{1}{q^2} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}} \right)^2 \sum_{l_1=1}^{q-1} |S(-l_1, -kc; q)|^2 \left(\frac{\sin \frac{Ll_1\pi}{q}}{\sin \frac{l_1\pi}{q}} \right)^2 \\
&\quad + \frac{L^2}{q^2} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}} \right)^2 |S(0, -kc; q)|^2.
\end{aligned}$$

Now recall Weil's bound on Kloosterman sums (see [10, Corollary 11.12] for example)

$$S(a, b; q) \ll (a, b, q)^{1/2} q^{1/2} d(q) \quad (2)$$

and

$$\begin{aligned}
T(N; q) &:= \sum_{k=1}^{q-1} \left(\frac{\sin \pi N k / q}{\sin \pi k / q} \right)^2 \\
&= \sum_{k=1}^{q-1} \sum_{d=-N}^N (N - |d|) e\left(\frac{dk}{q}\right) = \sum_{d=-N}^N (N - |d|) \sum_{k=1}^{q-1} e\left(\frac{dk}{q}\right) \\
&= \sum_{d=-N}^N (N - |d|) \sum_{k=1}^q e\left(\frac{dk}{q}\right) - N^2 = qN - N^2
\end{aligned}$$

for $0 \leq N \leq q$. As $T(N \pm q; q) = T(N; q)$, we have $|T(N; q)| \leq qN$ for all $N \geq 0$. Also, by grouping the sum according to the greatest common divisor of k and q ,

$$\sum_{k=1}^{q-1} \left(\frac{\sin \pi L k / q}{\sin \pi k / q} \right)^2 (k, q) = \sum_{d|q} d \sum_{\substack{k'=1 \\ (k', q/d)=1}}^{q/d-1} \left(\frac{\sin \frac{Lk'\pi}{q/d}}{\sin \frac{k'\pi}{q/d}} \right)^2 \leq \sum_{d|q} d \frac{q}{d} L = qLd(q). \quad (3)$$

Since $(c, q) = 1$, using (2) and (3), we have

$$\begin{aligned}
S &\ll \frac{d(q)^2}{q} \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}} \right)^2 \sum_{l_1=1}^{q-1} (l_1, q) \left(\frac{\sin \frac{Ll_1\pi}{q}}{\sin \frac{l_1\pi}{q}} \right)^2 + \frac{L^2}{q} d(q)^2 qLd(q) \\
&\leq Ld(q)^3 \sum_{k=1}^{q-1} \left(\frac{\sin \frac{Lk\pi}{q}}{\sin \frac{k\pi}{q}} \right)^2 + L^3 d(q)^3 \ll L^2 qd(q)^3
\end{aligned}$$

as $L \leq q$. This proves Theorem 1.

3 Corollaries 2 and 3

Proof of Corollary 2: Let $L = \lceil C_\epsilon q^{3/4+\epsilon} \rceil$. Suppose there are some integers a_0 and b_0 such that the congruence equation $q_1 q_2 \equiv c \pmod{q}$ has no solution with $a_0 < q_1 \leq a_0 + 2L$, $b_0 < q_2 \leq b_0 + 2L$. Then by Theorem 1, we have

$$L^2 \left(0 - \frac{L^2}{q}\right)^2 \ll L^2 q d(q)^3$$

as the congruence equation $q_1 q_2 \equiv c \pmod{q}$ has no solution with $a < q_1 \leq a + L$, $b < q_2 \leq b + L$ for all $a_0 \leq a \leq a_0 + L$ and $b_0 \leq b \leq b_0 + L$. The above inequality gives

$$\frac{L^6}{q^2} \ll L^2 q d(q)^3 \quad \text{or} \quad L^4 \ll q^3 d(q)^3.$$

This leads to $\lceil C_\epsilon q^{3/4+\epsilon} \rceil = L \ll q^{3/4} d(q)^{3/4}$ which is impossible if C_ϵ is large enough (using $d(q) \ll_\epsilon q^\epsilon$). Hence we have Corollary 2.

Proof of Corollary 3: Set $L := \lceil C_\epsilon q^{1/2+\epsilon} \rceil$. For $1 \leq x, y \leq q-1$ and $xy \equiv 1 \pmod{q}$, define the square and circle centered at (x, y) by

$$S_{(x,y)}(L) := \{(a, b) : x - L - 1 \leq a \leq x + L + 1, y - L - 1 \leq b \leq y + L + 1\},$$

and

$$C_{(x,y)}(L) := \{(a, b) : (a - x)^2 + (b - y)^2 \leq L^2\}.$$

If $\bigcup_{x,y=1, xy \equiv 1 \pmod{q}}^{q-1} S_{(x,y)}(L)$ covers the square $[0, q] \times [0, q]$, then the circles $\bigcup_{x,y=1, xy \equiv 1 \pmod{q}}^{q-1} C_{(x,y)}(\sqrt{2}(L+1))$ would cover the square $[0, q] \times [0, q]$ and we are done.

Consider $L < a, b < q - L$ (this is to avoid “wrap” around squares \pmod{q} when applying Theorem 1). Suppose (a, b) is not covered by $\bigcup_{x,y=1, xy \equiv 1 \pmod{q}}^{q-1} S_{(x,y)}(L)$. Then the square $([a], [a] + L) \times ([b], [b] + L)$ does not contain any solution to $q_1 q_2 \equiv 1 \pmod{q}$ with $q_1 \in ([a], [a] + L]$ and $q_2 \in ([b], [b] + L]$ for otherwise the square $S_{(q_1, q_2)}(L)$ would contain (a, b) . Call $([a], [b])$ a “bad” lattice point and let N be the number of such “bad” lattice points. Theorem 1 tells us that $N \frac{L^4}{q^2} \ll L^2 q d(q)^3$. Hence $N \ll \frac{q^3 d(q)^3}{L^2} = o(q^2)$ for C_ϵ large enough (using $d(q) \ll_\epsilon q^{\epsilon/2}$). Since every (a, b) with $L < a, b < q - L$ not covered is associated to some “bad” lattice points, the area not covered by $\bigcup_{x,y=1, xy \equiv 1 \pmod{q}}^{q-1} S_{(x,y)}(L)$ and hence $\bigcup_{x,y=1, xy \equiv 1 \pmod{q}}^{q-1} C_{(x,y)}(\sqrt{2}(L+1))$ must be $o(q^2)$ and we have Corollary 3 since the area outside of $L < a, b < q - L$ is $O(qL) = o(q^2)$.

4 Theorem 3 and Corollary 4

Proof of Theorem 3: Let

$$\chi_a(m) = \begin{cases} 1 & \text{if } a < m \leq a + L \pmod{q}, \\ 0 & \text{otherwise.} \end{cases}$$

and its Fourier coefficients

$$\hat{\chi}_a(n) = \frac{1}{q} \sum_{l \pmod{q}} \chi_a(l) e\left(-\frac{ln}{q}\right) = \frac{1}{q} \sum_{a < l \leq a+L} e\left(-\frac{ln}{q}\right).$$

Then $\chi_a(m) = \sum_{k \pmod{q}} \hat{\chi}_a(k) e\left(\frac{km}{q}\right)$ as its finite Fourier series.

For $(c, q) = 1$, we have

$$S = \sum_{a_1=1}^q \cdots \sum_{a_t=1}^q \left| \sum_{\substack{q_1=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \cdots \sum_{\substack{q_t=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \chi_{a_1}(q_1) \cdots \chi_{a_t}(q_t) - \left(\frac{L}{q}\right)^t \phi(q)^{t-1} \right|^2.$$

Since $\hat{\chi}_a(q) = L/q$,

$$\sum_{\substack{q_1=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \cdots \sum_{\substack{q_t=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \hat{\chi}_{a_1}(q) \cdots \hat{\chi}_{a_t}(q) = \left(\frac{L}{q}\right)^t \phi(q)^{t-1}.$$

Thus by using the finite Fourier series of $\chi_a(m)$, we have

$$S = \sum_{a_1=1}^q \cdots \sum_{a_t=1}^q \left| \sum_{\substack{q_1=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \cdots \sum_{\substack{q_t=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \sum_{1 \leq k_1, \dots, k_t \leq q}^* \hat{\chi}_{a_1}(k_1) e\left(\frac{k_1 q_1}{q}\right) \cdots \hat{\chi}_{a_t}(k_t) e\left(\frac{k_t q_t}{q}\right) \right|^2$$

where \sum' means summing over those numbers that are relatively prime to q and the $*$ means that we sum over all possible k 's except $k_1 = \cdots = k_t = q$. Expanding things out, we have

$$\begin{aligned} S &= \sum_{a_1, \dots, a_t} \sum'_{\substack{q_1, \dots, q_t \\ q_1 \cdots q_t \equiv c \pmod{q}}} \sum'_{\substack{q'_1, \dots, q'_t \\ q'_1 \cdots q'_t \equiv c \pmod{q}}} \sum_{k_1, \dots, k_t}^* \sum_{k'_1, \dots, k'_t}^* \\ &\quad \hat{\chi}_{a_1}(k_1) \overline{\hat{\chi}_{a_1}(k'_1)} e\left(\frac{k_1 q_1}{q}\right) e\left(-\frac{k'_1 q'_1}{q}\right) \cdots \hat{\chi}_{a_t}(k_t) \overline{\hat{\chi}_{a_t}(k'_t)} e\left(\frac{k_t q_t}{q}\right) e\left(-\frac{k'_t q'_t}{q}\right). \end{aligned}$$

Observe that

$$\begin{aligned} &\sum_{a_1=1}^q \hat{\chi}_{a_1}(k_1) \overline{\hat{\chi}_{a_1}(k'_1)} \\ &= \frac{1}{q^2} \sum_{a_1=1}^q \sum_{a_1 < l_1 \leq a_1+L} e\left(-\frac{l_1 k_1}{q}\right) \sum_{a_1 < l'_1 \leq a_1+L} e\left(\frac{l'_1 k'_1}{q}\right) \\ &= \frac{1}{q^2} \sum_{a_1=1}^q \sum_{a_1 < l_1 \leq a_1+L} e\left(\frac{l_1(k'_1 - k_1)}{q}\right) \sum_{a_1 - l_1 < s_1 \leq a_1+L-l_1} e\left(\frac{s_1 k'_1}{q}\right) \\ &= \frac{1}{q^2} \sum_{l_1=1}^q e\left(\frac{l_1(k'_1 - k_1)}{q}\right) \sum_{-L \leq s_1 \leq L} (L - |s_1|) e\left(\frac{s_1 k'_1}{q}\right) \\ &= \begin{cases} \frac{1}{q} \sum_{-L \leq s_1 \leq L} (L - |s_1|) e\left(\frac{s_1 k'_1}{q}\right), & \text{if } k_1 \equiv k'_1 \pmod{q}, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

by substituting $l'_1 = l_1 + s_1$ and moving the sum over a_1 inside. Thus

$$\begin{aligned}
S &= \frac{1}{q^t} \sum'_{q_1, \dots, q_t} \sum'_{q'_1, \dots, q'_t} \sum^*_{k_1, \dots, k_t} e\left(\frac{k_1(q_1 - q'_1)}{q}\right) \cdots e\left(\frac{k_t(q_t - q'_t)}{q}\right) \\
&\quad \times \left(\frac{\sin \frac{\pi L k_1}{q}}{\sin \frac{\pi k_1}{q}}\right)^2 \cdots \left(\frac{\sin \frac{\pi L k_t}{q}}{\sin \frac{\pi k_t}{q}}\right)^2 \\
&= \frac{1}{q^t} \sum^*_{k_1, \dots, k_t} \left(\frac{\sin \frac{\pi L k_1}{q}}{\sin \frac{\pi k_1}{q}}\right)^2 \cdots \left(\frac{\sin \frac{\pi L k_t}{q}}{\sin \frac{\pi k_t}{q}}\right)^2 \left| \sum'_{q_1, \dots, q_t} e\left(\frac{k_1 q_1 + \dots + k_t q_t}{q}\right) \right|^2 \\
&\quad \quad \quad q_1 \cdots q_t \equiv c \pmod{q}
\end{aligned}$$

by Fejér kernel formula $(\frac{\sin \pi N x}{\sin \pi x})^2 = \sum_{j=-N}^N (N - |j|) e(jx)$. Here we use the convention that $(\frac{\sin \pi L k/q}{\sin \pi k/q})^2 = L^2$ if $k = q$. The sum over the q 's is a hyper-Kloosterman sum. Now recall Weinstein's version [14] of Deligne's result on hyper-Kloosterman sums:

$$\left| \sum'_{q_1, \dots, q_t} e\left(\frac{k_1 q_1 + \dots + k_t q_t}{q}\right) \right| \leq C_q q^{\frac{t-1}{2}} t^{\omega(q)} (k_1, k_t, q)^{\frac{1}{2}} \cdots (k_{t-1}, k_t, q)^{\frac{1}{2}}$$

$q_1 \cdots q_t \equiv c \pmod{q}$

where $C_q = 1$ if q is odd and $C_q = 2^{(t+1)/2}$ if q is even, and (a, b, c) stands for the greatest common divisor of a , b and c .

Using the above bound, we have

$$S \leq \frac{C_q^2 t^{2\omega(q)}}{q} \sum^*_{k_1, \dots, k_t} \left(\frac{\sin \frac{\pi L k_1}{q}}{\sin \frac{\pi k_1}{q}}\right)^2 \cdots \left(\frac{\sin \frac{\pi L k_t}{q}}{\sin \frac{\pi k_t}{q}}\right)^2 (k_1, k_t, q) \cdots (k_{t-1}, k_t, q).$$

We estimate the above sum according to whether i of the k 's are equal to q with $0 \leq i < t$. If $i = 0$, then it is bounded by

$$\left[\sum_{k=1}^{q-1} \left(\frac{\sin \frac{\pi L k}{q}}{\sin \frac{\pi k}{q}}\right)^2 (k, q) \right]^t \leq q^t L^t d(q)^t$$

by (3). If $i > 0$, there are two cases depending on $k_t = q$ or $k_t \neq q$.

When $k_t = q$, there are $i - 1$ of the k 's that can be q . So we have the bound

$$\binom{t-1}{i-1} q^{i-1} L^{2i} \left[\sum_{k=1}^{q-1} \left(\frac{\sin \frac{\pi L k}{q}}{\sin \frac{\pi k}{q}}\right)^2 (k, q) \right]^{t-i} \leq \binom{t-1}{i-1} q^{t-1} L^{t+i} d(q)^t$$

where $q^{i-1} L^{2i}$ comes from the $i - 1$ such k 's together with k_t .

When $k_t \neq q$, we have the bound

$$\begin{aligned}
&\binom{t-1}{i} L^{2i} \left[\sum_{k=1}^{q-1} \left(\frac{\sin \frac{\pi L k}{q}}{\sin \frac{\pi k}{q}}\right)^2 (k, q) \right]^{t-i-1} \left[\sum_{k_t=1}^{q-1} \left(\frac{\sin \frac{\pi L k_t}{q}}{\sin \frac{\pi k_t}{q}}\right)^2 (k_t, q)^i \right] \\
&\leq \binom{t-1}{i} L^{2i} (q L d(q))^{t-i-1} q^i L d(q) \leq \binom{t-1}{i} q^{t-1} L^{t+i} d(q)^t
\end{aligned}$$

where L^{2i} comes from the i such k 's and they also contribute $(k_t, q)^i$ to the sum over k_t . Combining the above upper bounds, we have

$$S \leq \frac{C_q^2 t^{2\omega(q)}}{q} q^t L^t d(q)^t \left[1 + \frac{\sum_{i=1}^{t-1} \binom{t}{i} L^i}{q} \right] \leq C_q^2 t^{2\omega(q)} q^{t-1} L^t d(q)^t \left[1 + \frac{t(L+1)^{t-1}}{q} \right]$$

by $(L+1)^t - L^t \leq t(L+1)^{t-1}$. This proves Theorem 3.

Proof of Corollary 4: Recall

$$S = \sum_{a_1=1}^q \cdots \sum_{a_t=1}^q \left| \sum_{\substack{q_1=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \cdots \sum_{\substack{q_t=1 \\ q_1 \cdots q_t \equiv c \pmod{q}}}^q \chi_{a_1}(q_1) \cdots \chi_{a_t}(q_t) - \left(\frac{L}{q} \right)^t \phi(q)^{t-1} \right|^2$$

Let N be the number of tuples (a_1, \dots, a_t) such that $q_1 \cdots q_t \equiv c \pmod{q}$ has no solution with $a_1 < q_1 \leq a_1 + L, \dots, a_t < q_t \leq a_t + L$. Then by Theorem 3,

$$N \frac{L^{2t}}{q^2} \left(\frac{\phi(q)}{q} \right)^{2t-2} \leq C_q^2 t^{2\omega(q)} q^{t-1} L^t d(q)^t \left[1 + \frac{t(L+1)^{t-1}}{q} \right].$$

Since we are going to pick $L \leq q^{1/(t-1)} - 1$, we have

$$N \leq C_q^2 (1+t) t^{2\omega(q)} d(q)^t \left(\frac{q}{\phi(q)} \right)^{2t-2} q^t \frac{q}{L^t}$$

which is $o(q^t)$ if $L = q^{1/t+\epsilon}$. This gives Corollary 4.

5 Higher moment attack

In general, one expects that the error in

$$\sum'_{\substack{q_1 \in (a_1, a_1+L] \\ q_1 \cdots q_t \equiv c \pmod{q}}} \cdots \sum'_{\substack{q_t \in (a_t, a_t+L] \\ q_1 \cdots q_t \equiv c \pmod{q}}} 1 - \left(\frac{L}{q} \right)^t \phi(q)^{t-1}$$

is about the square root of the main term when $L \gg q^{1/t+\epsilon}$. Focusing on $t = 2$, we expect

$$\sum'_{\substack{q_1 \in (a_1, a_1+L] \\ q_1 q_2 \equiv c \pmod{q}}} \sum'_{\substack{q_2 \in (a_2, a_2+L] \\ q_1 q_2 \equiv c \pmod{q}}} 1 - \left(\frac{L}{q} \right)^2 \phi(q) \ll_{\epsilon} \frac{L}{q^{1/2-\epsilon}}.$$

Raising to the k -th power and summing over a_1, a_2 , we arrive at the following

Conjecture 1 *For any $\epsilon > 0$ and positive integer k ,*

$$\sum_{a_1=1}^q \sum_{a_2=1}^q \left| \sum'_{\substack{q_1 \in (a_1, a_1+L] \\ q_1 q_2 \equiv c \pmod{q}}} \sum'_{\substack{q_2 \in (a_2, a_2+L] \\ q_1 q_2 \equiv c \pmod{q}}} 1 - \left(\frac{L}{q} \right)^2 \phi(q) \right|^k \ll_{\epsilon, k} \frac{L^k}{q^{k/2-2-\epsilon}}$$

for $L \gg q^{1/2+\epsilon}$.

Theorems 1 and 3 show that Conjecture 1 is true for $k = 2$. Now we imitate the proof of Corollary 2. Suppose there are integers a_0 and b_0 such that the congruence equation $q_1 q_2 \equiv c \pmod{q}$ has no solution with $a_0 < q_1 \leq a_0 + 2L$, $b_0 < q_2 \leq b_0 + 2L$. Then by Conjecture 1, we have

$$L^2 \left| 0 - \frac{L^2}{q} \right|^k \ll_{\epsilon, k} \frac{L^k}{q^{k/2-2-\epsilon}}$$

as the congruence equation $q_1 q_2 \equiv c \pmod{q}$ has no solution with $a < q_1 \leq a + L$, $b < q_2 \leq b + L$ for all $a_0 \leq a \leq a_0 + L$ and $b_0 \leq b \leq b_0 + L$. The above inequality gives

$$L \ll_{\epsilon, k} q^{1/2+1/(k+2)+\epsilon}.$$

Consequently, if $L \gg_{\epsilon, k} q^{1/2+1/(k+2)+\epsilon}$, then $q_1 q_2 \equiv c \pmod{q}$ always has a solution with $a_1 < q_1 \leq a_1 + L$, $a_2 < q_2 \leq a_2 + L$ for any a_1, a_2 .

In particular, if Conjecture 1 is true for $k = 3$ or $k = 4$, then Question 1 is true for all $\epsilon > 1/5$ or $\epsilon > 1/6$ respectively. These are better than the currently best result. In fact, if Conjecture 1 is true for arbitrarily large k , we would settle Question 1 for all $\epsilon > 0$. So the next challenge is to prove Conjecture 1 say for $k = 4$ even with a slightly larger upper bound. This would be a major breakthrough!

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